Lecture 39

Saturday, April 9, 2022 11:11 PM

* Prayer
* Spiritual thought

We have learned line integral and surface integral.

$$\int_C f\,ds, \quad \underbrace{\int_C F\cdot dr}_{\text{circulation}}, \quad \iint_S f\,dS, \quad \underbrace{\iint_S \vec{F}\cdot d\vec{S}}_{\text{flux}}$$

orientation sensitive

There are theorems that help find integrals more quickly:

- Fundamental thm of Calculus
- Green's theorem
- Stokes' theorem
- Divergence theorem (Gauss-Ostragradsky)

Recall:

Fundamental theorem: $\int_C \nabla\phi\cdot dr = \phi(B)-\phi(A)$

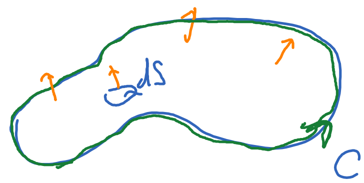
Green's thm: $\int_C F\cdot dr = \iint_R (Q_x - P_y)\,dA$

close, simple, positively oriented

* Stokes' theorem:

$$\int_C F \cdot dr = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

$\uparrow$ closed, simple, positively oriented w.r.t. S



Imagine walking along the curve. The surface has to be on the left

the direction from your feet to head is the direction of the normal vector of the surface.

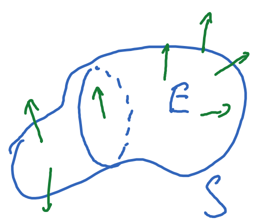
Why is this true?

$$\operatorname{curl} \vec{F} \cdot d\vec{S} = \underbrace{\operatorname{curl} \vec{F} \cdot \vec{n}\, dS}_{\text{circulation in } dS}$$

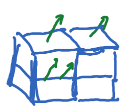
Sum of all of those local circulations gives a global circulation.

* Divergence thm: (also called Gauss-Ostrogradsky)

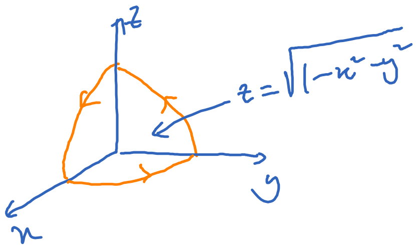
S is oriented outward.



$$\underbrace{\iint_S \vec{F} \cdot d\vec{S}}_{\text{total flux}} = \text{flux out of the solid} = \iiint_E \underbrace{\operatorname{div} \vec{F}}_{\text{flux density}} dV$$



Ex



$$\vec{F} = (-y, z, x)$$

Find $\int_C \vec{F} \cdot d\vec{r}$.

$$\int_C \vec{F} \cdot dr = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

∂_x	∂_y	∂_z
$-y$	z	x

$\operatorname{curl} F = (-1, -1, 1)$

Parametrization of the surface:

$$\begin{cases} x = \sin\phi\cos\theta \\ y = \sin\phi\sin\theta \\ z = \cos\phi \end{cases} \qquad \begin{matrix} 0 \le \theta \le \frac{\pi}{2} \\ 0 \le \phi \le \frac{\pi}{2} \end{matrix}$$

$$r_\phi = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi)$$

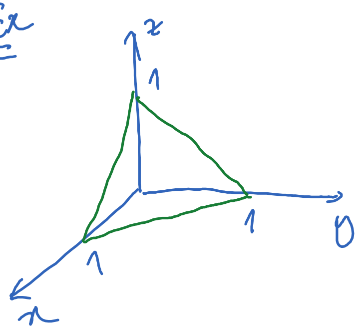
$$r_\theta = (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0)$$

$$r_\phi \times r_\theta = (\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \cos\phi\sin\phi) \leftarrow \text{correct orientation}$$

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_R (-1,1,1)\cdot(\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \cos\phi\sin\phi)\,dA$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} (-\sin^2\phi\cos\phi + \sin^2\phi\sin\theta + \cos\phi\sin\phi)\,d\phi\,d\theta = \ldots$$

Ex



S is made of 4 triangles, oriented outward.

$$\iint_S (xy, yz, zx)\cdot d\vec{S} = ?$$

Method 1: use parametrization

$$\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4}$$

Method 2: use Divergence theorem

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \nabla\cdot\vec{F}\, dV = \iiint_E (x+y+z)\, dV$$

$$\left[\nabla\cdot(xy+yz+zx) = y+z+x\right]$$

$$= \iint_R \int_0^{1-x-y} (x+y+z)\, dz\, dA = \cdots$$